

# Classification and Measurement of Multipartite Quantum Entanglements

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## Abstract

This paper presents a new measure of entanglement which can be employed for multipartite entangled systems. The classification of multipartite entangled systems based on this measure is considered. Two approaches to applying this measure to mixed quantum states are discussed.

# 1 Introduction

Entanglement has always been an important concept in quantum physics, thus there has been significant effort in quantifying entanglement. Among the many review articles concerning this task are [1] and [2]. Several measures have been developed for bipartite entanglement, in particular the entanglement of formation and negativity measures. However, the extension of these measures to tripartite or multipartite systems is a difficult problem [3], [5]. In [3], an attempt is made to extend the definition of maximally entangled systems to multipartite systems, while [5] provides a classification for entanglements in 3 qubit systems based on the extended form of the Schmidt decomposition for 3 qubits [4].

A major concern with all entanglement measures is the approach taken for mixed states. The idea of purifying and then measuring the entanglement was proposed in [7]. They introduced an entanglement measure based on purification of bipartite quantum systems, and then applied the entanglement of formation measure on the pure states. This approach solves the problem of classifying and measuring mixed state entanglements. However, there is a significant drawback with this approach, namely, the measured value for a separable mixed state and an entangled pure state may be the same. As an example, consider the separable mixed state

$$\rho = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10|]$$

and the pure entangled state

$$[|\psi\rangle = \frac{1}{\sqrt{2}}[|001\rangle + |110\rangle].$$

It is clear that the purified version of  $\rho$  is equal to  $|\psi\rangle$ , and therefore applying any entanglement measure on these states will produce the same outcome even though  $\rho$  is separable.

Two approaches are presented in this paper for the classification and measurement of multipartite entanglement. First, the concept of purifying an entangled system is extended to multipartite systems using the generalized Schmidt decomposition (GSD) proposed in [4] for ternary qubit (qutrit) systems and extended in [9] to multipartite systems of any dimension. The second approach is to decompose mixed states to their pure state components and then apply the measure on these components taking into consideration the probability of occurrence of each component. Both of these measurement techniques can be employed with multipartite systems of any dimension.

## 2 A New Entanglement Measure

In order to classify multipartite entangled systems, some definitions are required. An entanglement is called an  **$n$ -partite pure entanglement** when there is an entanglement between the  $n$  parties, but there is no entanglement if one of the parties is traced out. System 9 in Fig. 1 is an example of this case, which in general is given by  $\frac{1}{\sqrt{2}}(|0\dots 0\rangle + |1\dots 1\rangle)$ .

An  $n$ -partite system has a  **$p$ -mixed entanglement** if there exist  $p$  sets of  $n$ -partite local operations  $S_1, \dots, S_p$  with  $\epsilon_1^i \otimes \dots \otimes \epsilon_n^i \in S_i$  and there are members of these sets that when applied to the system preserve the  $m_i$ -partite pure entanglements. For such systems, tracing out one party does not break the entanglement between all of the parties. Note that the terms pure and mixed entanglements above do not denote pure and mixed states. Here we provide an example of what we mean by  $p$ -mixed entanglement. Consider the 2-mixed entanglement of a set of 5 parties ( $A, B, C, D$  and  $E$ ) shown in Fig. 2. The sets of local operations  $S_1$  and  $S_2$  with  $\epsilon_1^1 \otimes \dots \otimes \epsilon_5^1 \in S_1$  and  $\epsilon_1^2 \otimes \dots \otimes \epsilon_5^2 \in S_2$ , respectively, give

$$\begin{aligned}\epsilon_1^1 \otimes \dots \otimes \epsilon_5^1[|ABCDE\rangle_{\text{mixed entangled}}] &= |ABC\rangle_{\text{pure entangled}} \otimes |D\rangle \otimes |E\rangle \\ \epsilon_1^2 \otimes \dots \otimes \epsilon_5^2[|ABCDE\rangle] &= |ABCD\rangle_{\text{pure entangled}} \otimes |E\rangle\end{aligned}$$

As can be observed, there is at least one local operation in  $S_1$  which gives rise to a 3-partite pure entanglement between the parties A, B, and, C and also two separate states D and E. Further, there is a local operation in  $S_2$  which results in a 4-partite pure entanglement and one separate party.

A **fully entangled** multipartite system is defined as follows. A fully entangled  $n$ -partite system is one in which all possible sets of  $m$ -partite systems,  $m < n$ , are entangled. System 16 in fig. 1 is an example of a fully entangled system. The following proposition establishes the connection between **fully entangled** and **maximally entangled** systems.

**Proposition 2.1** *A maximally entangled system is fully entangled.*

**Proof.** Assume we have an  $n$ -partite system. According to [1], a maximally entangled system is a system from which all the other entangled and pure states can be produced via local operations. In addition, local operations are not able to create entanglement. Assuming that all pure  $m$ -partite entanglements are possible for an  $n$ -partite entanglement,  $m < n$ , then the maximally entangled  $n$ -partite system must be fully entangled in order to be able to produce all pure  $m$ -partite entangled systems. Establishing the existence of  $m$ -partite entanglements in an  $n$ -partite system is straightforward. An example of such an entanglement is

$$\frac{1}{\sqrt{2}}|0 \cdots 0\rangle_{n-m} \otimes (|0 \cdots 0\rangle_m + |1 \cdots 1\rangle_m)$$

□

Note that this proof does not guarantee the existence of a fully entangled  $n$ -partite system. However, it does imply that if there is no fully entangled  $n$ -partite system then a maximally entangled  $n$ -partite system cannot exist.

Fig. 1 shows all possible types of entanglement in a 3-partite system. There are 8 different classes of entanglement including the non-entangled system, namely  $\{1\}$ ,  $\{2, 3, 4\}$ ,  $\{5, 6, 7\}$ ,  $\{8\}$ ,  $\{9\}$ ,  $\{10, 11, 12\}$ ,  $\{13, 14, 15\}$ , and  $\{16\}$ . Note that systems 2, 3, and 4 are similar, but with different parties involved. Finding all possible entanglements for a  $n$ -partite system is a simple combinatorics problem. Some examples of entangled systems are classified below accordance to Fig. 1. Simple entanglement examples are given by Systems 2, 3 and 4 in Fig. 1. In these cases, two states are

maximally entangled but the other one is separate from them

$$\frac{1}{\sqrt{2}}|000\rangle + |011\rangle).$$

An example of a 3-partite pure entanglement is given by System 9 in Fig. 1, with

$$\frac{1}{\sqrt{2}}|000\rangle + |111\rangle).$$

All three parties are entangled, but if one is traced out the other two become disentangled. The last example of a 3-partite entanglement is given by System 8 in Fig. 1, with

$$\frac{1}{\sqrt{3}}|010\rangle + |100\rangle + |001\rangle).$$

The three 3 parties are again entangled, but in this case tracing out one of the parties does not eliminate the entanglement between the other two parties.

A mixed entangled state is simply a combination of one or more types of entanglements. Another means of dealing with mixed states is to purify them and then classify them according to the resulting pure states, as in Fig. 1 for 3-partite systems. Note that, as discussed in the Introduction, they will be entangled states after purification.

The above definitions and discussions provide some insight into the requirements for a proper measure of entanglement. For example, one can say

intuitively that the system

$$\frac{1}{\sqrt{3}}|010\rangle + |100\rangle + |001\rangle$$

is more entangled than the system

$$\frac{1}{\sqrt{2}}|000\rangle + |111\rangle.$$

Although both systems have all parties entangled, the first does not become disentangled when one of the parties is traced out. A good measure of entanglement must take this into account.

Let  $\rho^{1\dots n}$  be an  $n$ -partite entangled system in the Hilbert space  $H_1 \otimes \dots \otimes H_n$  (note that the dimensions  $\dim(H_i)$  need not be equal). If  $\{\lambda_i, |\psi_i\rangle\}$  are eigenvalues and eigenvectors of  $\rho^{1\dots n}$ , then a generic purification for the system is given by

$$|\psi_{\text{pure}}\rangle = \sum_i \sqrt{\lambda_i} |\psi_i\rangle |0^1 \dots 0^{n-1} i^n\rangle \quad (1)$$

so that an ancillary qubit  $|0\rangle$  is attached to each system except the last where  $|i\rangle$  is attached. All other purifications can be derived from (1) by applying local unitary transforms. From Theorem 3 in [9], there exists a local equivalent of  $|\psi_{\text{pure}}\rangle$ , say

$$|\psi_{\text{decompose}}\rangle = U_1 \otimes \dots \otimes U_n |\psi_{\text{pure}}\rangle,$$

where  $|\psi_{\text{decompose}}\rangle$  can be expressed in the form

$$\sum_{i_1 \dots i_n} C_{i_1 \dots i_n} |\psi_{i_1}^{(1)}\rangle \dots |\psi_{i_n}^{(n)}\rangle,$$

and  $\{|\psi_i^{(r)}\rangle\}$  is a fixed orthonormal basis for the state space  $H_r$ . The Ingarden-Urbanik (IU) entropy can be used to measure the entanglement, which gives

$$M(\rho) = S_{IU}(|\psi_{\text{decompose}}\rangle) = S_{IU}(U_1 \otimes \dots \otimes U_n |\psi_{\text{pure}}\rangle) = - \sum_{i_1 \dots i_n} |C_{i_1 \dots i_n}|^2 \log |C_{i_1 \dots i_n}|^2.$$

Note that  $|\psi_{\text{decompose}}\rangle$  is locally equivalent to  $|\psi_{\text{pure}}\rangle$  and therefore is just a purification of  $\rho^{1 \dots n}$ . The Schmidt number plays a crucial role in this measure, as the larger the number of terms in the generalized Schmidt decomposition (i.e., the greater the number of nonzero  $C_{i_1 \dots i_n}$ ), the larger the IU entropy. This dependency on the Schmidt number (generalized Schmidt number in this case), is a desirable feature for an entanglement measure because, as discussed in [8], the Schmidt number is related to the amount of entanglement.

## 2.1 Additivity

Consider two systems  $\rho$  and  $\sigma$  with  $n$  and  $m$  parties, respectively. The standard purification of  $\rho \otimes \sigma$  is  $|\psi_{\text{pure}}\rangle = \sum_{i,j}^{n,m} \sqrt{\lambda_i \delta_j} |\psi_i\rangle |\phi_j\rangle |0^1 \dots 0^{mn-1} i j^{mn}\rangle$ . Then the tensor product of the generalized Schmidt decomposition of  $\rho$  and



$\sigma$  is given by

$$\sum_{i_1 \dots i_n} C_{i_1 \dots i_n} |\psi_{i_1}^{(1)}\rangle \dots |\psi_{i_n}^{(n)}\rangle \otimes \sum_{j_1 \dots j_n} C'_{j_1 \dots j_n} |\psi_{j_1}^{(1)}\rangle \dots |\psi_{j_n}^{(n)}\rangle$$

Using the entanglement measure we have

$$M(\rho \otimes \sigma) = - \sum_{i_1 \dots i_n, j_1 \dots j_n} |C_{i_1 \dots i_n} C_{j_1 \dots j_n}|^2 \log |C_{i_1 \dots i_n} C_{j_1 \dots j_n}|^2 = M(\rho) + M(\sigma).$$

where  $\sum_{i_1 \dots i_n} |C_{i_1 \dots i_n}|^2 = 1$ . Therefore the measure is additive. From the above result, we also have that  $M(\rho \otimes \sigma) = M(\sigma \otimes \rho)$ .

## 2.2 Continuity

The continuity of the measure is considered for states with generalized Schmidt decompositions which are close. This is because two states  $\rho$  and  $\sigma$  may not be close with respect to the trace distance, but their local equivalents, which can be expressed in the form of Schmidt decompositions, can be close. Since our measure is based on these decompositions, we only require the following.

**Proposition 2.2** *If the trace distance for the generalized Schmidt decomposition of two arbitrary states  $\rho$  and  $\sigma$  is less than  $\epsilon$*

$$D(GSD(\rho), GSD(\sigma)) < \epsilon$$

*then*

$$|M(\rho) - M(\sigma)| < \epsilon \log(N)$$

where  $N$  is the dimension of the Hilbert spaces of  $\rho$  and  $\sigma$ .

**Proof.** This is just a special case of the Theorem in Fannes [6].  $\square$

Another measure is obtained by applying the GSD technique on the pure states and then computing  $S_{IU}$ . For the mixed states, instead of purifying and applying the measure for a state  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , the measure is computed as  $M(\rho) = S_{IU}(\sqrt{p_i}|\psi_i\rangle)$  where  $p_i$  and  $|\psi_i\rangle$  are eigenvalues and eigenvectors of  $\rho$ .

**Proposition 2.3** *The measure  $M[\cdot]$  given above is additive.*

**Proof.** The proof is similar to that for the additivity of the first measure based on purification, and therefore is omitted.  $\square$

As discussed in the Introduction, averaging over the pure components of the states has the advantage of properly measuring the separable mixed states, whereas the first measure may not be able to distinguish between pure entangled states and mixed separable states.

### 3 Conclusions

A new universal measure of entanglement was introduced. It is based on extending the idea of purification and then applying a measurement technique on the pure states. The additivity and continuity of this measure were examined. Measuring the pure components of a mixed state instead of employing purification was also considered.

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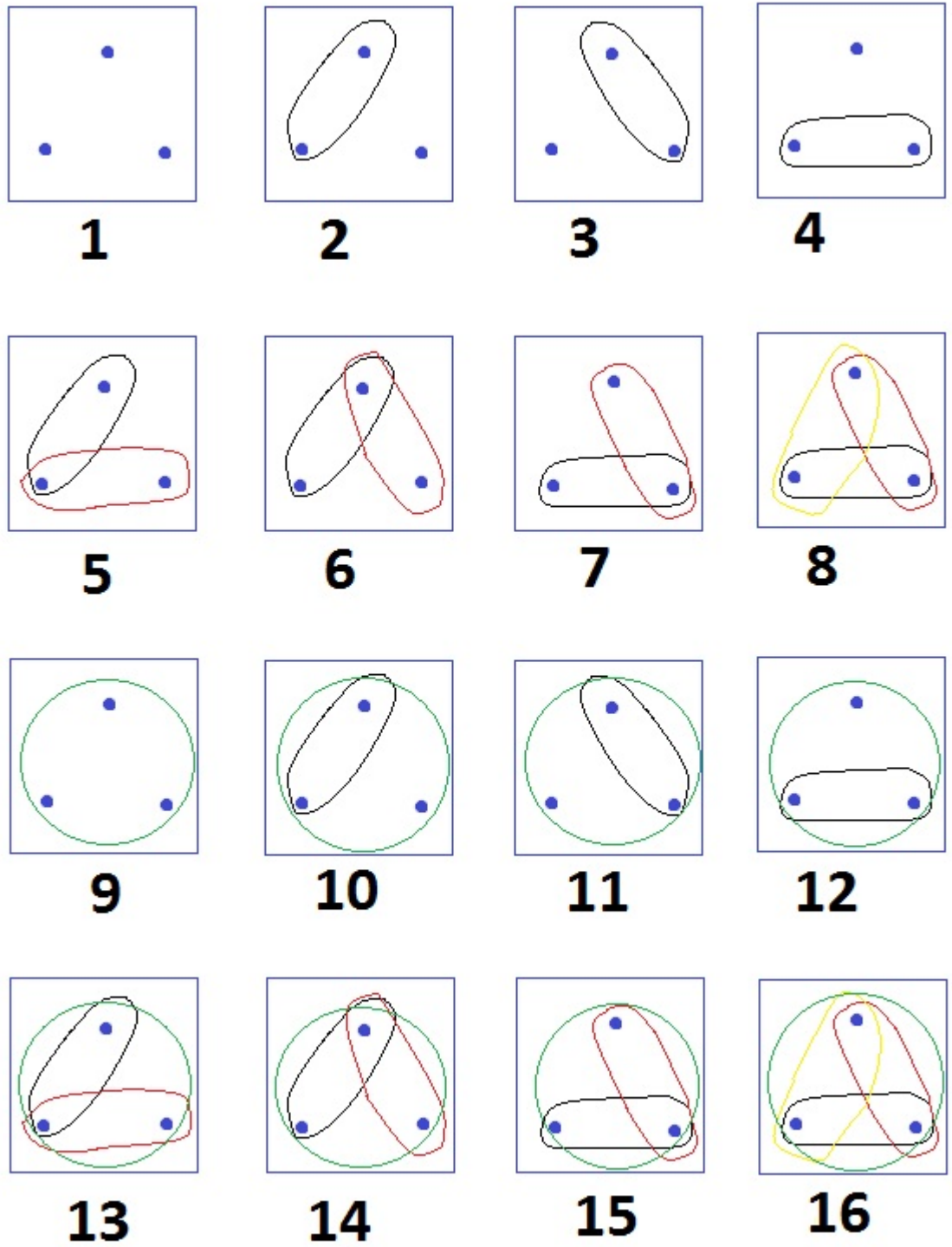


Figure 1: All possible entanglements for a pure 3-partite state. The outer square defines the borders of the isolated system, and each colored line around a set of parties is an entanglement.

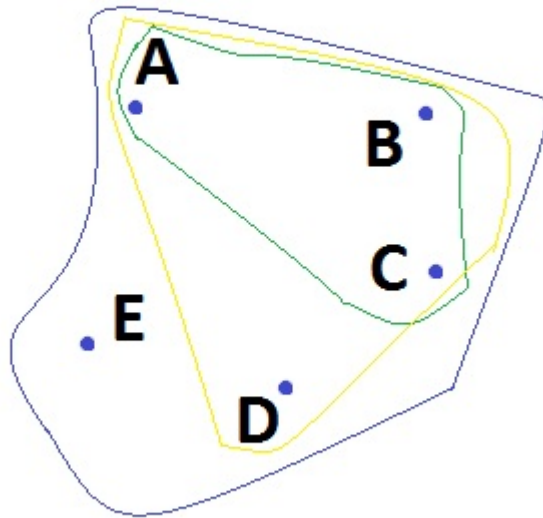


Figure 2: An example of a 2-mixed entangled system.